One-Dimensional, Steady-State Conduction without Thermal Energy Generation

Lecture 5 - Chapter Three
Sections 3.1 through 3.4

Methodology of a Conduction Analysis

Specify appropriate form of the heat equation.
Solve for the temperature distribution.
Apply Fourier’s law to determine the heat flux.

Simplest Case: One-Dimensional, Steady-State Conduction with No Thermal Energy Generation.

Common Geometries:
The Plane Wall: Described in rectangular ($x$) coordinate. Area perpendicular to direction of heat transfer is constant (independent of $x$).
The Tube Wall: Radial conduction through tube wall.
The Spherical Shell: Radial conduction through shell wall.
Consider a plane wall between two fluids of different temperature:

Assuming steady-state conditions and no internal heat generation \((i.e., \dot{q} = 0)\), then the 1-D heat conduction equation reduces to:

\[
\frac{d}{dx}\left(k \cdot A \frac{dT}{dx}\right) = 0
\]

For constant \(k\) and \(A\):

\[
\frac{d^2T}{dx^2} = 0
\]

The implications are:

- Heat flux \((q_x)\) is independent of \(x\).
- Heat rate \((q_x)\) is independent of \(x\).

Boundary Conditions: \(T(0) = T_{s,1}, \ T(L) = T_{s,2}\)

Temperature Distribution shown for constant \(k\)

The Plane Wall

\[
1-D \text{ heat conduction equation for steady-state conditions and no internal heat generation } (i.e., \dot{q} = 0),
\]

is:

\[
\frac{d^2T}{dx^2} = 0 \text{, for constant } k \text{ and } A
\]

Integrate twice to get \(T(x)\):

\[
T(x) = \int \int \frac{d^2T}{dx^2} \, dx \, dx = \int \left( \frac{dT}{dx} + C_1 \right) \, dx
\]

\[
\therefore \ T(x) = C_1 \cdot x + C_2
\]

for Boundary Conditions: \(T(0) = T_{s,1}, \ T(L) = T_{s,2}\)

\[
\therefore \text{ at } x = 0, \ T(x) = T_{s,1}, \ \text{and } C_2 = T_{s,1}
\]

\[
\text{at } x = L, \ T(x) = T_{s,2}, \ \text{and } \therefore T_{s,2} = C_1 \cdot L + C_2 = C_1 \cdot L + T_{s,1}
\]

which gives \(\therefore C_1 = \frac{T_{s,2} - T_{s,1}}{L}\), Substituting the values for \(C_1\) and \(C_2\) in to exp. for \(T(x)\):

\[
\therefore \ T(x) = \frac{x}{L} (T_{s,2} - T_{s,1}) + T_{s,1} \text{ and apply Fourier's Law to get heat transfer, } q_x
\]
Plane Wall (cont.)

Heat Flux and Heat Rate:

\[ q_x = -kA \frac{dT}{dx} = \frac{kA}{L} \left( T_{s,1} - T_{s,2} \right) \]

\[ q''_x = -k \frac{dT}{dx} = \frac{k}{L} \left( T_{s,1} - T_{s,2} \right) \]

Thermal Resistances \( R_t = \frac{\Delta T}{q} \) and Thermal Circuits

Conduction in a plane wall: \( R_{t,\text{cond}} = \frac{L}{kA} \)

Convection: \( R_{t,\text{conv}} = \frac{1}{hA} \)

Thermal circuit for plane wall with adjoining fluids:

\[ R_{tot} = \frac{1}{h_1A} + \frac{L}{kA} + \frac{1}{h_2A} \]

\[ q_x = \frac{T_{x,1} - T_{x,2}}{R_{tot}} \]

Plane Wall (cont.)

Thermal Resistance for Unit Surface Area:

\[ R''_{t,\text{cond}} = \frac{L}{k} \quad R''_{t,\text{conv}} = \frac{1}{h} \]

Units: \( R_t \leftrightarrow \text{K/W} \quad R''_t \leftrightarrow \text{m}^2 \cdot \text{K/W} \)

Radiation Resistance:

\[ R_{t,\text{rad}} = \frac{1}{h_rA} \quad R''_{t,\text{rad}} = \frac{1}{h_r} \]

\[ h_r = \varepsilon \sigma \left( T_s + T_{\text{sur}} \right) \left( T_s^2 + T_{\text{sur}}^2 \right) \]
Composite Wall with Negligible Contact Resistance:

\[
\frac{1}{h_A A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_A A} = \sum R_i
\]  

Overall Heat Transfer Coefficient \((U)\):

A modified form of Newton’s Law of Cooling to encompass multiple resistances to heat transfer.

\[
q_x = U A \Delta T_{overall}
\]  

\[
R_{tot} = \frac{1}{U A}
\]
Contact Resistance:

\[ R^*_{t,c} = \frac{T_A - T_B}{\frac{q'_x}{T_T}} \]

\[ R_{t,c} = \frac{R^*_{t,c}}{A_c} \]

Values depend on: Materials A and B, surface finishes, interstitial conditions, and contact pressure (Tables 3.1 and 3.2)

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**Table 3.1** Thermal contact resistance for (a) metallic interfaces under vacuum conditions and (b) aluminum interface (10-μm surface roughness, 10^5 N/m^2) with different interfacial fluids [1]

<table>
<thead>
<tr>
<th>(a) Vacuum Interface</th>
<th>(b) Interfacial Fluid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contact pressure</td>
<td>100 kN/m^2 10,000 kN/m^2</td>
</tr>
<tr>
<td>Stainless steel</td>
<td>6–25 0.7–4.0</td>
</tr>
<tr>
<td>Copper</td>
<td>1–10 0.1–0.5</td>
</tr>
<tr>
<td>Magnesium</td>
<td>1.5–3.5 0.2–0.4</td>
</tr>
<tr>
<td>Aluminum</td>
<td>1.5–5.0 0.2–0.4</td>
</tr>
</tbody>
</table>

Air 2.75
Helium 1.05
Hydrogen 0.720
Silicone oil 0.525
Glycerine 0.265

tbl_03_01
Electrical Analogy of Thermal Circuits

To solve a parallel resistance network like that shown opposite, we can reduce the network to and equivalent resistance.

For electrical circuits:

\[ R_{\text{par}} = \frac{R_2 \times R_3}{R_2 + R_3} \]

and

\[ R_{\text{Total}} = R_1 + R_{\text{par}} + R_4 \]

\[ I = \frac{\Delta V}{R_{\text{Total}}} \]

For thermal circuits:

\[ q_x = \frac{\Delta T}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t} \]
Series – Parallel Composite Wall:

Note: error increases as there is a departure from the one-dimensional conditions if $k_F \neq k_G$.

Circuits based on assumption of isothermal surfaces normal to $x$ direction or adiabatic surfaces parallel to $x$ direction provide approximations for $q_x$.

Example: Parallel resistances
These IR photos show that the heat flow through the built-up walls is more complex than indicated by a simple parallel-resistance, 1-D thermal model. However, it will still provide a rough estimate of the average, wall heat transfer rate. The images clearly show “thermal bridging” by the wall studs.

Approximate Solution

It is common practice to estimate the heat loss rate through an insulated stud-wall by area-weighting the projected area of the studs and insulation cavity to arrive at an average thermal resistance or “U-factor”

Note: + (normal to the direction of average heat conduction)

\[
U_{\text{average}} = (U_{\text{insul}} \times \frac{A_{\text{insul}}}{A_{\text{wall}}}) + (U_{\text{stud}} \times \frac{A_{\text{studs}}}{A_{\text{wall}}})
\]

\[
U_{\text{average}} = (U_{\text{insul}} \times F_{\text{area insul}}) + (U_{\text{ stud}} \times F_{\text{area studs}})
\]

\[
q_{\text{wall}} = U_{\text{average}} \cdot A_{\text{wall}} (T_{\text{indoor}} - T_{\text{outdoor}})
\]

\[
q_{\text{wall}} / A_{\text{wall}} = q''_{\text{wall}} = U A_{\text{wall}} (T_{\text{indoor}} - T_{\text{outdoor}})
\]

Here \( U A_{\text{wall}} \) is the average area weighted "U-factor" or "U-value" for the wall, (i.e., \( U A_{\text{wall}} = U_{\text{average}} \))

\[
q_{\text{wall}} = (U A_{\text{wall}} \cdot A_{\text{wall}} (T_{\text{indoor}} - T_{\text{outdoor}})
\]
Alternative Wall Design

Example of resistance network with both a radiative and convective boundary.

\[ q \rightarrow T_i \rightarrow T_s \rightarrow T_{\text{sur}} \rightarrow T_{\infty} \]

- \( T_i = 35^\circ \text{C} \)
- \( k_{sf} = 0.3 \text{ W/m-K} \)
- \( L_{sf} = 3 \text{ mm} \)
- \( T_{\text{sur}} = 10^\circ \text{C} \)
- \( e = 0.95 \)
- \( k_{ins} = 0.014 \text{ W/m-K} \)
- \( L_{ins} \)
- \( h = 2 \text{ W/m}^2\text{-K (Air)} \)
- \( h = 200 \text{ W/m}^2\text{-K (Water)} \)
Other 1-D Geometries

Heat Equation:

\[ \frac{1}{r} \frac{d}{dr} \left( kr \frac{dT}{dr} \right) = 0 \]

What does the form of the heat equation tell us about the variation of \( q_r \) with \( r \) in the wall?

Is the foregoing conclusion consistent with the energy conservation requirement?

How does \( q_r \) vary with \( r \)?

Temperature Distribution for Constant \( k \):

\[ T(r) = \frac{T_{s,1} - T_{s,2}}{\ln \left( \frac{r_2}{r} \right)} \ln \left( \frac{r_2}{r} \right) + T_{s,2} \]

**Figure 3.6** Hollow cylinder with convective surface conditions.

The Tube Wall

Heat Equation:

\[ \frac{1}{r} \frac{d}{dr} \left( kr \frac{dT}{dr} \right) = 0 \]

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Temperature Distribution for Constant \( k \):

\[ T(r) = \frac{T_{s,1} - T_{s,2}}{\ln \left( \frac{r_2}{r} \right)} \ln \left( \frac{r_2}{r} \right) + T_{s,2} \]
Heat Flux and Heat Rate:

\[ q_r^* = -k \frac{dT}{dr} = \frac{k}{r \ln(r_2 / r_1)} (T_{s,1} - T_{s,2}) \]

\[ q_r' = 2\pi r q_r^* = \frac{2\pi k}{\ln(r_2 / r_1)} (T_{s,1} - T_{s,2}) \]

\[ q_r = 2\pi r L q_r^* = \frac{2\pi L k}{\ln(r_2 / r_1)} (T_{s,1} - T_{s,2}) \]  \hspace{1cm} (3.27)

Conduction Resistance:

\[ R_{t,\text{cond}} = \frac{\ln(r_2 / r_1)}{2\pi L k} \quad \text{Units ↔ K/W} \] \hspace{1cm} (3.28)

\[ R_{t,\text{cond}}' = \frac{\ln(r_2 / r_1)}{2\pi k} \quad \text{Units ↔ m·K/W} \]

Why is it inappropriate to base the thermal resistance on a unit surface area?

---

**Composite Wall with Negligible Contact Resistance**

![Composite Wall with Negligible Contact Resistance](image)

**Figure 3.7** Temperature distribution for a composite cylindrical wall.
Composite Wall with Negligible Contact Resistance

\[ q_r = \frac{T_{r,1} - T_{r,4}}{R_{tot}} = UA(T_{r,1} - T_{r,4}) \]

Note that

\[ UA = R_{tot}^{-1} \]

is a constant independent of radius.

But, \( U \) itself is tied to specification of an interface.

\[ U_i = \left( A_i R_{tot} \right)^{-1} \quad (3.32) \]

Spherical Shell

Heat Equation

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0 \]

What does the form of the heat equation tell us about the variation of \( q_r \) with \( r \)? Is this result consistent with conservation of energy?

How does \( q_r \) vary with \( r \)?

Temperature Distribution for Constant \( k \):

\[ T(r) = T_{s,1} - \left( T_{s,1} - T_{s,2} \right) \frac{1 - \left( r_1/r \right)}{1 - \left( r_1/r_2 \right)} \]
Heat flux, Heat Rate and Thermal Resistance:

\[
q''_r = -k \frac{dT}{dr} = \frac{k}{r^2 \left[ \left( \frac{1}{\eta_1} \right) - \left( \frac{1}{\eta_2} \right) \right]} (T_{s,1} - T_{s,2})
\]

\[
q_r = 4\pi r^2 q''_r = \frac{4\pi k}{\left( \frac{1}{\eta_1} \right) - \left( \frac{1}{\eta_2} \right)} (T_{s,1} - T_{s,2})
\]

\[
R_{t,cond} = \frac{\left( \frac{1}{\eta_1} \right) - \left( \frac{1}{\eta_2} \right)}{4\pi k}
\]

Composite Shell:

\[
q_r = \frac{\Delta T_{overall}}{R_{tot}} = UA \Delta T_{overall}
\]

\[
UA = R_{tot}^{-1} \leftrightarrow \text{Constant}
\]

\[
U_i = (A_i R_{tot})^{-1} \leftrightarrow \text{Depends on } A_i
\]

Critical radius for insulation: see example 3.5 in Text

\[
r_{cr} = \frac{k}{h}
\]
Summary

![Summary Diagram](image)

### Table 3.3: One-dimensional, steady-state solutions to the heat equation with no generation

<table>
<thead>
<tr>
<th></th>
<th>Plane Wall</th>
<th>Cylindrical Wall</th>
<th>Spherical Wall</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heat equation</td>
<td>$\frac{d^2T}{dx^2} = 0$</td>
<td>$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0$</td>
<td>$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0$</td>
</tr>
<tr>
<td>Temperature distribution</td>
<td>$T_{s,1} - \Delta T \frac{x}{L}$</td>
<td>$T_{s,2} + \Delta T \frac{\ln (r_2/r_2)}{\ln (r_1/r_2)}$</td>
<td>$T_{s,1} - \Delta T \left[ \frac{1 - (r_1/r_2)}{1 - (r_1/r_2)} \right]$</td>
</tr>
<tr>
<td>Heat flux ($q^*$)</td>
<td>$k \frac{\Delta T}{L}$</td>
<td>$k \frac{\Delta T}{r \ln (r_2/r_1)}$</td>
<td>$k \frac{\Delta T}{r^2 [(1/r_1) - (1/r_2)]}$</td>
</tr>
<tr>
<td>Heat rate ($q$)</td>
<td>$kA \frac{\Delta T}{L}$</td>
<td>$\frac{2\pi Lk \Delta T}{\ln (r_2/r_1)}$</td>
<td>$\frac{4\pi k \Delta T}{(1/r_1) - (1/r_2)}$</td>
</tr>
<tr>
<td>Thermal resistance ($R_{t,cond}$)</td>
<td>$\frac{L}{kA}$</td>
<td>$\frac{\ln (r_2/r_1)}{2\pi Lk}$</td>
<td>$\frac{(1/r_1) - (1/r_2)}{4\pi k}$</td>
</tr>
</tbody>
</table>

*The critical radius of insulation is $r_{cr} = k/l$ for the cylinder and $r_{cr} = 2k/\ell$ for the sphere.
Problem 3.23: Assessment of thermal barrier coating (TBC) for protection of turbine blades. Determine maximum blade temperature with and without TBC.

ASSUMPTIONS: (1) One-dimensional, steady-state conduction in a composite plane wall, (2) Constant properties, (3) Negligible radiation

Analysis: For a unit area, the total thermal resistance with the TBC is

\[
R_{\text{tot},w}^* = h_0^{-1} + (L/k)_{Zr} + R_{t,c}^* + (L/k)_{In} + h_i^{-1}
\]

\[
R_{\text{tot},w}^* = \left(10^{-3} + 3.85 \times 10^{-4} + 10^{-4} + 2 \times 10^{-4} + 2 \times 10^{-3}\right) \text{m}^2 \cdot \text{K/W} = 3.69 \times 10^{-3} \text{m}^2 \cdot \text{K/W}
\]

With a heat flux of

\[
q_w^* = \frac{T_{s,o} - T_{s,i}}{R_{\text{tot},w}^*} = \frac{1300 \text{K}}{3.69 \times 10^{-3} \text{m}^2 \cdot \text{K/W}} = 3.52 \times 10^5 \text{W/m}^2
\]

the inner and outer surface temperatures of the Inconel are

\[
T_{s,i(w)} = T_{s,i} + (q_w^*/h_i) = 400 \text{K} + \left(3.52 \times 10^5 \text{W/m}^2 / 500 \text{W/m}^2 \cdot \text{K} / \text{W}\right) = 1104 \text{K}
\]

\[
T_{s,o(w)} = T_{s,o} + \left[(1/h_i) + (L/k)_{In}\right]q_w^* = 400 \text{K} + \left(2 \times 10^{-3} + 2 \times 10^{-4}\right) \text{m}^2 \cdot \text{K/W} \left(3.52 \times 10^5 \text{W/m}^2 \cdot \text{m}^2\right) = 1174 \text{K}
\]
Without the TBC,

\[
R_{\text{tot,wo}}^* = h_0^{-1} + (L/k)_{\text{In}} + h_i^{-1} = 3.20 \times 10^{-3} \, \text{m}^2 \cdot \text{K}/\text{W}
\]

\[
q_{\text{wo}}^* = \left( T_{\infty,0} - T_{x,i} \right) / R_{\text{tot,wo}}^* = 4.06 \times 10^4 \, \text{W/m}^2
\]

The inner and outer surface temperatures of the Inconel are then

\[
T_{s,i(\text{wo})} = T_{x,i} + \left( q_{\text{wo}}^* / h_i \right) = 1212 \, \text{K}
\]

\[
T_{s,o(\text{wo})} = T_{x,i} + \left[ (1 / h_i) + (L/k)_{\text{In}} \right] q_{\text{wo}}^* = 1293 \, \text{K}
\]

Use of the TBC facilitates operation of the Inconel below \( T_{\text{max}} = 1250 \, \text{K} \).

**COMMENTS:** Since the durability of the TBC decreases with increasing temperature, which increases with increasing thickness, limits to its thickness are associated with reliability considerations.

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**Problem: Radioactive Waste Decay**

**Problem 3.62:** Suitability of a composite spherical shell for storing radioactive wastes in oceanic waters.

**SCHEMATIC:**

![Diagram of composite spherical shell with lead and stainless steel layers](Image)

**ASSUMPTIONS:** (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties at 300K, (4) Negligible contact resistance.

**PROPERTIES:** Table A-1, Lead: \( k = 35.3 \, \text{W/m-K} \), MP = 601K; St.St.: 15.1 W/m-K.

**ANALYSIS:** From the thermal circuit, it follows that

\[
q = \frac{T_i - T_{x,i}}{R_{\text{tot}}} = q \left[ \frac{4}{3} \pi r_i \right]
\]
Problem: Radioactive Waste Decay

The thermal resistances are:

\[ R_{\text{Pb}} = \left[ \frac{1}{(4\pi \times 35.3 \text{ W/m} \cdot \text{K})} \right] \left[ \frac{1}{0.25 \text{m}} - \frac{1}{0.30 \text{m}} \right] = 0.00150 \text{ K/W} \]

\[ R_{\text{St.St.}} = \left[ \frac{1}{(4\pi \times 15.1 \text{ W/m} \cdot \text{K})} \right] \left[ \frac{1}{0.30 \text{m}} - \frac{1}{0.31 \text{m}} \right] = 0.000567 \text{ K/W} \]

\[ R_{\text{conv}} = \left[ \frac{1}{4\pi \times 0.31^2 \text{ m}^2 \times 500 \text{ W/m}^2 \cdot \text{K}} \right] = 0.00166 \text{ K/W} \]

\[ R_{\text{tot}} = 0.00372 \text{ K/W}. \]

The heat rate is then
\[ q = 5 \times 10^5 \text{ W/m}^3 \left( \frac{4\pi}{3} \right) (0.25 \text{m})^3 = 32,725 \text{ W} \]

and the inner surface temperature is
\[ T_i = T_{\infty} + R_{\text{tot}} q = 283K + 0.00372K/W \times 32,725 \text{ W} = 405 \text{ K} < \text{MP} = 601K. \]

Hence, from the thermal standpoint, the proposal is adequate.

**COMMENTS:** In fabrication, attention should be given to maintaining a good thermal contact. A protective outer coating should be applied to prevent long term corrosion of the stainless steel.